1. (a) Suppose that T is a projection.

Then $v \in \operatorname{im} T$ iff $Tv = v$.

For, if $v \in \operatorname{im} T$, then $v = Tw$ for some $w \in V$,
so then $Tv = T^2 w = Tw = v$
since $T^2 = T$.

The reverse implication is trivial.

Now $\ker T \cap \operatorname{im} T = \{0\}$,
for if $v \in \operatorname{im} T$, then $Tv = v$ as above,
and if $v \in \ker T$, then $Tv = 0$,
so $v = 0$.

Finally if $v \in V$, then $v = Tv + v - Tv$;
$Tv \in \operatorname{im} T$ and $v - Tv \in \ker T$
since
$$T(v - Tv) = Tv - T^2 v = Tv - Tv = 0.$$

So $V = \ker T \oplus \operatorname{im} T$ as required. [3 marks]

Now every element of $\ker T$ is an eigenvector with eigenvalue 0, and we have already seen that every element of $\operatorname{im} T$ is an eigenvector with eigenvalue 1. So there is a spanning set of eigenvectors & therefore a basis of eigenvectors. So T is diagonalisable.

[2 marks]

The above argument shows that there are no eigenvalues other than 0 and 1; so does the observation that since $T^2=T$, $T^2-T=0$, so $m_T(x) \mid x^2-x = x(x-1)$. ~~[2 marks]~~

The characteristic polynomial is $x^k(1-x)^\ell$, where $k = \dim \ker T$ and $\ell = \dim \operatorname{im} T$. [1 mark]

The minimum polynomial is

$$\begin{cases} x & \text{if } \ell = 0 \quad (\text{so } T=0) \\ 1-x & \text{if } k=0 \quad (\text{so } T=I) \\ x(1-x) & \text{otherwise.} \end{cases}$$

[2 marks]

(This is all on the problem sheets.)

$m_{\alpha I+\beta T}(x)$ is $(x-\alpha)$, if $T=0$, $(x-\alpha-\beta)$, if $T=I$, & $(x-\alpha)(x-\alpha-\beta)$ o/w. [2 marks]

(b) (i) $E+F$ is a projection iff

$$(E+F)^2 = E+F.$$

Now $(E+F)^2 = E^2+EF+FE+F^2$

$= E + EF + FE + F$ since E and F are projections,

[3 marks]

so $(E+F)^2 = E+F$

iff $EF+FE=0$, that is $EF=-FE$.

(ii). If $\mathbb{F}$ does not have characteristic 2, then the above implies that $0 = E(EF+FE) = E^2F + EFE$

$= EF + EFE,$

while $0 = (EF + FE)E = EFE + FE^2$
$= EFE + FE$,

so that $EF = FE$.

Since $EF = -FE$,

$2EF = 0$ so $EF = 0$ since $2 \neq 0$.

Hence also $FE = 0$.

Conversely if $EF = FE = 0$, then $EF = -FE$ so by (i) $E + F$ is a projection.

[4 marks]

(iii) In $(\mathbb{Z}_2)^3$, consider the three matrices

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

A, B and C all represent projections, and $C = A + B$, while $AB \neq 0$.

[3 marks]

(Familiar stuff, from past papers)

(c). The same three matrices, over a field not of characteristic 2, give a counterexample since A, B and C commute, $ABC = 0$,

but $A + B + C = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ which is not the matrix of a projection since 2 is an eigenvalue.

(New) [5 marks]

2(a) T is diagonalisable iff $m_T(x)$ is a product of distinct (monic) linear factors. (1)

(Bookwork) [2 marks]

$$\chi_A(x) = \begin{vmatrix} x & -1 & 0 \\ -6 & x & 2 \\ -4 & -6 & x+3 \end{vmatrix}$$

[or similarly with all terms multiplied by -1]

$$= x \begin{vmatrix} x & 2 \\ -6 & x+3 \end{vmatrix} + \begin{vmatrix} -6 & 2 \\ -4 & x+3 \end{vmatrix}$$

$$= x\Big(x(x+3)+12\Big) + (-6)(x+3) + 8$$

$$= x^3 + 3x^2 + 12x - 6x - 18 + 8$$

$$= x^3 + 3x^2 + 6x - 10.$$

By inspection 1 is a root. Then

$$x^3 + 3x^2 + 6x - 10$$
$$= (x-1)(x^2+4x+10),$$

and the roots of this are 1 and

$$\frac{-2 \pm \sqrt{16-40}}{2} = -1 \pm \sqrt{-6},$$

in any field not of characteristic 2 in which -6 has a square root; if the field is not of characteristic 2 and -6 has no square root, then $x^2+4x+10$ is irreducible.

2

(i). A is diagonalisable over $\mathbb{C}$ because $\chi_A(x)$ is a product of distinct linear factors and so by the Cayley-Hamilton Theorem $m_A(x)$ must be also.

[Or: A has 3 distinct eigenvalues and is 3×3, so it is diagonalisable.]

[2 marks]

(ii). Over $\mathbb{R}$, $x^2+4x+10$ is irreducible. Since $A \neq I$, $m_A(x) \neq x-1$, so $x^2+4x+10 \mid m_A(x)$, so A is not diagonalisable. [2 marks]

[Or: laboriously check that A has only a 1-dimensional eigenspace; or: use that every irreducible factor of $\chi_A(x)$ is a factor of $m_A(x)$.]

~~(iii). By the same reasoning, A is not diagonalisable over $\mathbb{Q}$.~~

(iii). Over $\mathbb{Z}_3$, $-6=0$, so $\chi_A(x)=(x-1)^3$. Since $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \neq I$, $A-I \neq 0$ so $m_A(x) \neq x-1$. So $m_A(x)$ has a repeated linear factor so A is not diagonalisable over $\mathbb{Z}_3$.

[2 marks]

(iv) In $\mathbb{Z}_2$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

and $\chi_A(x) = x^3 + 3x^2 + 6x - 10$

$= x^2(x+1)$.

It is clear that $A(A+I) \neq 0$,

so $m_A(x) \nmid x(x+1)$,

so $m_A(x)$ has a repeated linear factor

& A is not diagonalisable.

[2 marks].

(b). Primary Decomposition Theorem:

Suppose that $m_T(x) = p(x)q(x)$, where $p(x)$ & $q(x)$ are coprime.

Then $V = \ker p(T) \oplus \ker q(T)$,

$m_{T|\ker p(T)}(x) = p(x)$, and $m_{T|\ker q(T)}(x) = q(x)$.

(Bookwork) [2 marks]

It is easy from this to prove that if $m_T(x) = p_1(x)p_2(x)\cdots p_r(x)$, where the $p_i(x)$ are mutually coprime, then

$$V = \ker p_1(T) \oplus \ker p_2(T) \oplus \dots$$
$$\dots \oplus \ker p_r(T),$$

and $m_{T|\ker p_i(T)}(x) = p_i(x)$.

If T is diagonalisable, then $m_T(x)$ is a product of distinct linear ~~transformati~~ factors. Write

$$m_T(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_r).$$

Then by PDT,

$$V = \bigoplus_{i=1}^{r} \ker(T-\lambda_i I)$$, since the $x-\lambda_i$ are coprime,

and so $V = \bigoplus_{i=1}^{r} V_{\lambda_i}$.

This is equivalent to the given form since if $\lambda \notin \{\lambda_1, \dots \lambda_r\}$, then $\ker(T-\lambda I) = \{0\}$.

[3 marks]

Suppose that S & T commute, and that $v \in V_\lambda$.

Then $TSv = STv = S(\lambda v) = \lambda Sv$.

So $Sv \in V_\lambda$ as required.

Now S is diagonalisable, so $m_S(x)$ is a product of distinct linear factors; clearly $m_{S|V_\lambda}(x) \mid m_S(x)$ so $m_{S|V_\lambda}(x)$ is a product of distinct linear factors so $S|_{V_\lambda}$ is diagonalisable.

Let $\mathcal{B}_\lambda$ be a basis for ~~S~~ V_λ consisting of eigenvectors of $S|_{V_\lambda}$.

Then if $\mathcal{B} = \bigcup_\lambda \mathcal{B}_\lambda$, then $\mathcal{B}$ is a basis for V all of whose elements are eigenvectors of both S & T. [3 marks]

Now suppose that $\mathcal{B}$ is a basis with respect to which ${}_{\mathcal{B}}[S]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ are diagonal.

Then clearly these two matrices commute, so so do S and T. [2 marks]
(familiar stuff)

(c). Suppose that S_1, S_2 and S_3 are diagonalisable. For each $\lambda, \mu \in \mathbb{F}$, let

$$V_\mu = \ker(S_3 - \mu I),$$

and $V_{\lambda,\mu} = \ker(S_2 - \lambda I) \cap \ker(S_3 - \mu I)$.

Exactly as above, ~~there~~

$$V = \bigoplus_{\lambda,\mu} V_{\lambda,\mu}.$$

As above, S_1 commutes with S_2 & S_3 so ~~fixes~~ $\ker(S_2 - \lambda I)$ and $\ker(S_3 - \mu I)$ are S_1-invariant and so so is $V_{\lambda,\mu}$.

As above let $\mathcal{B}_{\lambda,\mu}$ be a basis of eigenvectors for $S_1|_{V_{\lambda,\mu}}$ for $V_{\lambda,\mu}$.

Then $\mathcal{B} = \bigcup_{\lambda,\mu} \mathcal{B}_{\lambda,\mu}$ is a basis for V consisting of vectors which are simultaneously eigenvectors of S_1, S_2 and S_3.

(New) [5 marks].

3.(a).(i). e_i' is defined so that

$$e_i'(e_j) = \delta_{ij}. \quad [2 \text{ marks}]$$

(ii). Define $\Phi : V \to V''$ so that for all $v \in V$ & $f \in V'$,

$$\Phi(v)(f) = f(v).$$

(α) $\Phi(v) \in V''$:

Suppose that $f, g \in V'$ and $\alpha, \beta \in \mathbb{C}$.

Then $\Phi(v)(\alpha f + \beta g)$

$= (\alpha f + \beta g)(v)$ by definition of Φ

$= \alpha f(v) + \beta g(v)$ by definition of the vector space operations on V'

$= \alpha \Phi(v)(f) + \beta \Phi(v)(g)$ by definition of Φ.

So $\Phi(v)$ is linear. Since $\operatorname{ran} \Phi(v) \subseteq \mathbb{C}$, $\Phi(v) \in V''$.

(β) Φ is linear:

Suppose that ~~$z \in V$,~~ $u, v \in V$, $\alpha, \beta \in \mathbb{C}$, $f \in V'$.

Then $\Phi(\alpha u + \beta v)(f) = f(\alpha u + \beta v)$ by definition of Φ

$= \alpha f(u) + \beta f(v)$ since $f \in V'$ & so is linear

$= \alpha \Phi(u)(f) + \beta \Phi(v)(f)$ by definition of Φ

$= (\alpha \Phi(u) + \beta \Phi(v))(f)$ by definition of the v. sp. ops on V''.

So the functions $\Phi(\alpha u + \beta v)$ and $\alpha \Phi(u) + \beta \Phi(u)$ are equal.

So Φ is linear.

(γ). Φ is ~~onto~~ 1-1:

Suppose $\boxed{\cancel{v \in \ker \Phi}}$ $v \neq 0$.

Say $v = \sum_{i=1}^{n} \alpha_i e_i$, and $\alpha_i \neq 0$.

Then $e_i'(v) = \alpha_i \neq 0$. (*)

Hence $\Phi(v)(e_i') \neq 0$.

Hence $\ker \Phi$ is trivial so Φ is 1-1.

[Alternative to (*): if $v \neq 0$, extend v

to a basis $\{v, v_2 \dots v_n\}$, form a dual basis $\{f_1, f_2 \dots f_n\}$; then $f_1(v) \neq 0$.]

(δ). Now $\phi: V \to V''$ is a 1-1 linear transformation, and $\dim V'' = \dim V' = \dim V$, so ϕ is onto.

So ϕ is an ~~homom~~ isomorphism.

(Bookwork) [~~3~~5 marks]

(iii). Let $u = \sum_{i=1}^{n} \alpha_i e_i$ & $v = \sum_{i=1}^{n} \beta_i e_i$.

$$\text{Then } \varphi(u, v) = \tau(v)(u)$$

$$= \sum_{i=1}^{n} \overline{\beta_i}\, e_i' \left(\sum_{i=1}^{n} \alpha_i e_i \right)$$

$$= \sum_{i=1}^{n} \overline{\beta_i}\, \alpha_i \qquad \phi$$

φ is positive definite:

$$\varphi(u, u) = \sum_{i=1}^{n} \overline{\alpha_i}\, \alpha_i = \sum_{i=1}^{n} |\alpha_i|^2$$

which is ≥ 0 & $= 0$ iff $\alpha_i = 0$ for all i.

φ is linear in the first term:

Obvious.

φ is complex-conjugate symmetric:

Clear.

And, $\varphi(e_i, e_j) = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{o/w,} \end{cases}$

so φ is indeed an inner product wrt to which $\{e_1 \ldots e_n\}$ is orthonormal.

[Or, from ⊛ spot that (V, φ) is isomorphic to $(\mathbb{C}^n, \langle , \rangle)$, so φ is the innerproduct required.]

standard inner product

(New?) [5 marks]

(b)(i) Suppose $v \in V \setminus \{0\}$ and $Tv = \lambda v$.

Then $\lambda \langle v, v \rangle = \langle \lambda v, v \rangle$

$= \langle Tv, v \rangle$

$= \langle v, Tv \rangle$ by self-adjointness

$= \overline{\langle Tv, v \rangle}$

$= \overline{\langle \lambda v, v \rangle}$

(b) i. Suppose v is an eigenvector of T with eigenvalue λ.

Then
$$\begin{aligned}\lambda\langle v,v\rangle &= \langle Tv, v\rangle \\ &= \langle v, -Tv\rangle \quad \text{since } T^* = -T \\ &= \langle v, -\lambda v\rangle \\ &= \overline{\langle -\lambda v, v\rangle} \\ &= \overline{-\lambda\langle v,v\rangle} \\ &= \overline{-\lambda}\,\overline{\langle v,v\rangle} \\ &= \overline{-\lambda}\langle v,v\rangle.\end{aligned}$$

Since $v \neq 0$, $\lambda = \overline{-\lambda}$ so λ is purely imaginary. [3 marks]

ii. Suppose U is T-invariant & $v \in U^\perp$.
Then for all $u \in U$, $Tu \in U$, so
$$\langle Tu, v\rangle = 0$$
so $\langle u, Tv\rangle = 0$

so $Tv \perp u$.

Hence $Tv \in U^\perp$. [2 marks]

iii. We prove this by induction on $\dim V$.
If $\dim V = 0$ or 1 it is trivial.

Now suppose $\dim V > 1$.
By the Fundamental Theorem of Algebra T has an eigenvalue, λ say, which must have an eigenvector, which we label e_1.
Now $\{e_1\}^\perp$ is T-invariant by part ii., so by the inductive hypothesis there is a basis for $\{e_1\}^\perp$ consisting of eigenvectors of $T|_{\{e_1\}^\perp}$. Refer to this as $\{e_2 \cdots e_n\}$.
Then $\{e_1 \cdots e_n\}$ is a basis of eigenvectors of T.
[~~3~~ 4 marks]

(These three parts are modifications of bookwork arguments about self-adjoint linear transformations.)

iv. Suppose that A is a real antisymmetric matrix. Then considered as an element of $M_{n\times n}(\mathbb{C})$, it is diagonalisable, so its minimum polynomial is a product of distinct linear factors.
Now if $p(x)$ is any element of $\mathbb{C}[x]$,
$\overline{p(A)} = \bar{p}(\bar{A}) = \bar{p}(A)$, since A is real.
Hence ~~p~~ $\cancel{m_A(x A) = m_A}$
$\overline{m_A}(A) = 0$, so since $\overline{m_A}(x)$ is monic & of the same degree as $m_A(x)$, they must

be equal.

So $m_A(x)$ has real coefficients.

If now α is any root of $m_A(x)$,

$m_A(\alpha) = 0$ so $\overline{m_A(\alpha)} = 0$ so $m_A(\bar{\alpha}) = 0$.

So $\bar{\alpha}$ is also a root.

Also by i. all roots come in complex conjugate pairs.

Hence $m_A(x)$ has one of the two forms

$$x(x - i\alpha_1)(x + i\alpha_1)\dots(x - i\alpha_r)(x + i\alpha_r)$$

if 0 is a root

$$\& \quad (x - i\alpha_1)(x + i\alpha_1)\dots(x - i\alpha_r)(x + i\alpha_r)$$

if not.

[6 marks]

(New, but should be easy to see.)

~~v.~~ $\cancel{\frac{1}{\alpha} Tv = \frac{1}{\alpha} T\left(\frac{1}{\alpha} Tu\right) = \frac{1}{\alpha^2} T^2 u}$

$\cancel{= \frac{1}{\alpha^2}(-\alpha^2 u)}$ ~~since~~ $\cancel{u \in \ker(T^2 + \alpha^2 I^2)}$

$\cancel{= -u.}$

$\cancel{\langle u, v\rangle = \langle u, \frac{1}{\alpha} Tu\rangle}$

$\cancel{= \frac{1}{\alpha}\langle u, Tu\rangle}$

$\cancel{= \frac{1}{\alpha}\langle -Tu, u\rangle}$ ~~since~~

Q4

(a) (i) Autonomous means that there is no t-dependence in X or Y.
The nullclines are the curves given by $X(x,y)=0$ and $Y(x,y)=0$.
A critical point of the system is a point in the phase plane where $X=Y=0$.

[3]

(ii) A common way to analyse the solution of the system near a critical point at (a,b) is to linearise about that point.
Suppose $P=(a,b)$ is a critical point for (3), so that
$X(a,b)=0=Y(a,b)$.
We linearise by setting $x=a+\xi(t)$; $y=b+\eta(t)$
where ξ and η are small.

Then $$\frac{dx}{dt}=\dot{\xi}=X(a+\xi,b+\eta)=X(a,b)+\xi\frac{\partial X}{\partial x}\Big|_{(a,b)}+\eta\frac{\partial X}{\partial y}\Big|_{(a,b)}+\text{h.o.t}$$

$$\frac{dy}{dt}=\dot{\eta}=Y(a+\xi,b+\eta)=Y(a,b)+\xi\frac{\partial Y}{\partial x}\Big|_{(a,b)}+\eta\frac{\partial Y}{\partial y}\Big|_{(a,b)}+\text{h.o.t}$$

Now $X(a,b)=Y(a,b)=0$, and neglecting higher order terms, we may rewrite as

$$\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$ where

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}\frac{\partial X}{\partial x}\Big|_{(a,b)} & \frac{\partial X}{\partial y}\Big|_{(a,b)}\\ \frac{\partial Y}{\partial x}\Big|_{(a,b)} & \frac{\partial Y}{\partial y}\Big|_{(a,b)}\end{pmatrix}$$

Let $\underline{Z}(t)=\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix} \Rightarrow \dot{\underline{Z}}=\underline{M}\,\underline{Z}$

Solve via eigenvalues and eigenvectors as follows:
$\underline{Z}_0 e^{\lambda t}$ is a solution, with constant vector $\underline{Z}_0$ and constant scalar λ if $\lambda\underline{Z}_0=M\underline{Z}_0$

$\Rightarrow \underline{Z}_0$ is an eigenvector with eigenvalue λ.

If $\lambda_1\neq\lambda_2$ is $\underline{Z}=c_1\underline{Z}_1e^{\lambda_1 t}+c_2\underline{Z}_2e^{\lambda_2 t}$

If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ then

(i) If $M - \lambda I = 0 \Rightarrow \underline{z}(t) = \underline{c}\, e^{\lambda t}$

(ii) if $M - \lambda I \neq 0$ $\exists$ constant vector $\underline{z}_1$ with

$$\underline{z}_0 = (M - \lambda I)\underline{z}_1 \neq 0$$

but $(M - \lambda I)\underline{z}_0 = (M - \lambda I)^2 \underline{z}_1 = 0$

Solution is then $c_1 \underline{z}_1 + (c_0 + c_1 t)\underline{z}_0 \; e^{\lambda t}$.

Not mention now made re stability.

~~Have made no comments re stability even though this is expressly asked about in the question~~ [6]

A! Question has now been reworded.

Add to solution: If $Re(\lambda) > 0$ critical point is unstable

(b) $\dot{x} = x(1-y)(1-x-y)$

$\dot{y} = y(2-xy)$

Null clines are given by the curves $x(1-y)(1-x-y) = 0$

and $y(2-xy) = 0$.

ie $x = 0$, $y = 1$, $y = 1-x$; $y = 0$ and $y = 2/x$.

would a sketch be helpful? Sketch added later in solution pt (iii)

seems a relatively easy → [5] [2]

5 marks. Changed

i) critical points where $\Rightarrow x = 0,\ y = 1,\ y = 1-x$

$x(1-y)(1-x-y) = 0$ (+)

and $y(2-xy) = 0$

If $x = 0$ (+) $\Rightarrow y = 0$ $(0,0)$ critical pt.

if $y = 1$ (+) $\Rightarrow x = 2$ $(2,1)$ "

if $y = 1-x$ (+) $\Rightarrow 2 - x(1-x) = 0$ no root.

$\Rightarrow x^2 - x + 2 = 0$

$$x = \frac{1 \pm \sqrt{1-8}}{2}$$

$(1,0)$ critical pt.

or $1 - x = 0 \Rightarrow x = 1, y = 0$ [3]

$$M = \begin{pmatrix} (1-y)(1-x-y) - x(1-y) & -x(1-x-y) - x(1-y) \\ -y^2 & (2-xy) - xy \end{pmatrix}$$

~~[3]~~

[1]

$$M\Big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$(1-\lambda)(2-\lambda)=0$

$\lambda = 1, 2$

$\Rightarrow$ unstable node

$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$$M\Big|_{(2,1)} = \begin{pmatrix} 0 & -2(1-2-1) \\ -1 & 2-2-2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -1 & -2 \end{pmatrix}$$

$$(-\lambda)(-2-\lambda)+4=0$$

$$\lambda^2+2\lambda+4=0 \qquad \lambda = \frac{-2 \pm \sqrt{4-16}}{2}$$

$\Rightarrow$ stable spiral.

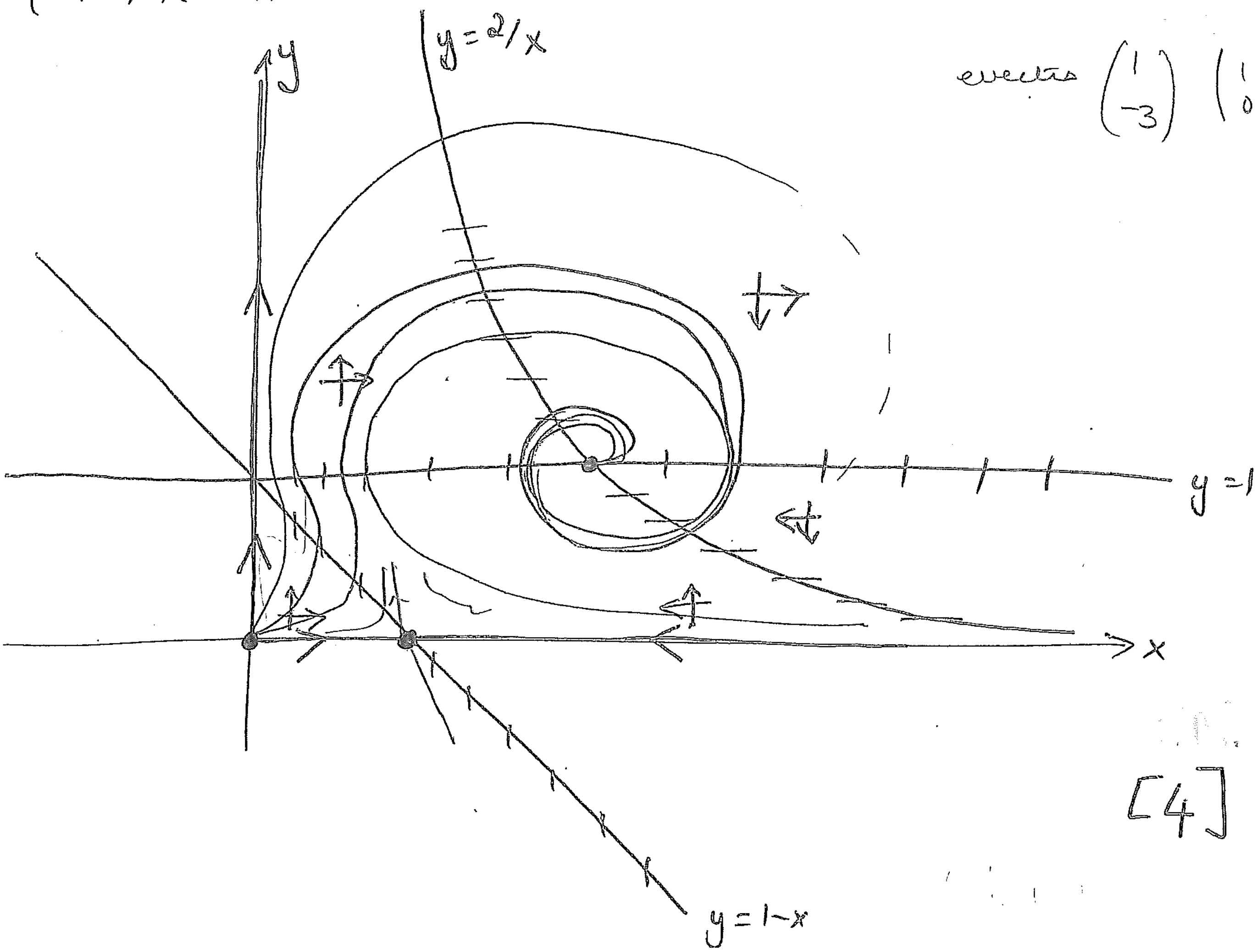
$$M\Big|_{(1,0)} = \begin{pmatrix} -1 & -1\cancel{(1-1)}-1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$$

[6]

$(-1-\lambda)(2-\lambda)=0 \quad \lambda = 2, -1 \quad \Rightarrow$ saddle.

evectrs $\begin{pmatrix}1\\-3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

ii)



[4]

(a) Q5

$$-x^2\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-x\frac{\partial u}{\partial x}=4$$

(i) $a=-x^2$
$b=0$
$c=1$
$\Rightarrow ac=-x^2<0=b \Rightarrow$ hyperbolic everywhere except $x=0$ $\boxed{2}$ ~~[1]~~ [S]

(ii) $-x^2\lambda^2+1=0 \qquad \lambda=\pm\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x},\ \frac{dy}{dx}=-\frac{1}{x}$ [4] [S]

$y=\ln|x|+\text{const};\ y=-\ln|x|+\text{const}$

$\Rightarrow$ characteristic variables as $\xi=y+\ln|x|;\ \eta=y-\ln|x|$.

$\xi_x=\frac{1}{x},\ \xi_y=1,\ \eta_x=-\frac{1}{x},\ \eta_y=1$

$u_x=u_\xi\xi_x+u_\eta\eta_x$
$=\frac{1}{x}u_\xi-\frac{1}{x}u_\eta$

$u_{xx}=-\frac{1}{x^2}u_\xi+\frac{1}{x}u_{\xi\xi}\xi_x+\frac{1}{x}u_{\xi\eta}\eta_x+\frac{1}{x^2}u_\eta-\frac{1}{x}u_{\eta\xi}\xi_x$
$-\frac{1}{x}u_{\eta\eta}\eta_x$

$u_{xx}=-\frac{1}{x^2}u_\xi+\frac{1}{x^2}u_{\xi\xi}-\frac{1}{x^2}u_{\xi\eta}+\frac{1}{x^2}u_\eta-\frac{1}{x^2}u_{\eta\xi}$
$+\frac{1}{x^2}u_{\eta\eta}$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi+u_\eta$

$u_{yy}=u_{\xi\xi}+2u_{\eta\xi}+u_{\eta\eta}$

$\Rightarrow -x^2\left[\frac{1}{x^2}\left(u_{\eta\eta}+u_{\xi\xi}-2u_{\eta\xi}-u_\xi+u_\eta\right)\right]$
$+u_{\xi\xi}+2u_{\eta\xi}+u_{\eta\eta}-\frac{x}{x}(u_\xi-u_\eta)=4$

$4u_{\xi\eta}=4 \qquad u_{\xi\eta}=1$ [4] [M]
$\Rightarrow u_\xi=\eta+f(\xi)$
$u=\eta\xi+g(\xi)+h(\eta)$

Hence $u=(y+\ln|x|)(y-\ln|x|)+g(y+\ln|x|)+h(y-\ln|x|)$ [3] [N/S]

3b. $u_{xx} + u_{yy} = f(x,y)$ (*)

Suppose first that $f < 0$ in D.
If u has an interior minimum at some point (x_0, y_0) in D, then the following conditions must be satisfied at (x_0, y_0):

$$u_x = u_y = 0, \quad u_{xx} \geq 0, \quad u_{yy} \geq 0$$

But f is strictly negative, and so (*) $\Rightarrow$ impossible (for both u_{xx} and u_{yy} to be non-negative (they sum to f)). Hence u cannot have an interior minimum within D, so it must attain its minimum on the bdy ∂D.

Suppose now that we have $f \leq 0$ in D. Perturb u to get a function v which satisfies Poisson's equation with RHS strictly negative, so can apply first part of the proof.

Consider $v(x,y) = u(x,y) - \frac{\varepsilon}{4}(x^2 + y^2)$ where ε is +ve constant.

Then $v_{xx} + v_{yy} = u_{xx} + u_{yy} - \varepsilon = f - \varepsilon < 0$
in D. So using the result just proved, v attains its minimum on ∂D.

Now suppose the minimum value of u on ∂D is M and the minimum value of $(x^2 + y^2)$ on ∂D is R^2. Then the minimum value of v on ∂D (and thus throughout D) is $M - \frac{\varepsilon}{4} R^2$

$\Rightarrow u - \frac{\varepsilon}{4}(x^2 + y^2) = v \geq M - \frac{\varepsilon}{4} R^2$ holds $\forall (x,y) \in D$

Letting $\varepsilon \to 0 \Rightarrow u \geq M$ throughout D.
$\Rightarrow u$ attains its minimum value on ∂D. $\boxed{6}$ ~~[7]~~

[03]

(ii) If $f = 0$ then since $f \le 0$, u attains its minimum on ∂D.

Now suppose $f \ge 0$

Now replace $u \to -u'$

$$\Rightarrow u'_{xx} + u'_{yy} = -f(x,y) = g(x,y)$$

where $g(x,y) \le 0$

Use above result $\Rightarrow u' = -u$ attains its minimum value on ∂D $\Rightarrow$ u attains its maximum value on ∂D.

So If $f = 0$, u attains both its maximum + minimum values on ∂D. [2] [B]

iii) $v \ge u$ on ∂D

Let $w(x,y) = v(x,y) - u(x,y)$

$w_{xx} + w_{yy} = 0$ ie $f = 0$ $\Rightarrow$ w attains its maximum and minimum value on ∂D

Since $v(x,y) - u(x,y) \ge 0$ on ∂D

$$\Rightarrow 0 \le \min_{(x,y) \in \partial D} v(x,y) - u(x,y) \le w(x,y)$$

$\Rightarrow$ $w \ge 0$ throughout D $\Rightarrow$ $v \ge u$ everywhere in D. [4] [N]

Q6

(i) Let $y_1 = y(x)$

$y_2 = y'(x)$

$$\Rightarrow \quad y_1' = y_2$$
$$y_2' = F(x, y_1, y_2)$$

Are there enough easy marks in this question? Bookbook focus

Let $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\Rightarrow \quad \underline{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix} = \begin{pmatrix} y_2 \\ F(x, y_1, y_2) \end{pmatrix}$$

$$\underline{y}(a) = \begin{pmatrix} y_1(a) \\ y_2(a) \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} = \underline{b}$$

[2]

Hence as given with $f_1(x, \underline{y}(x)) = y_2(x)$.

(ii) Rewrite equation (2) as an integral equation

$$\underline{y}(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))\, ds \qquad (*)$$

, where by integral we mean that we integrate componentwise.

Now define $(T\underline{y})(x) = \underline{b} + \int_a^x \underline{f}(s, \underline{y}(s))\, ds$

So we can write $(*)$ as a fixed point problem in $\mathcal{C}_\eta$

$$\underline{y} = T\underline{y}.$$

Proof Prove that $T : \mathcal{C}_\eta \to \mathcal{C}_\eta$ and is a contraction.

Take $\eta \leq h$.

rom properties of integration, $Ty(x) \in C(|x-a| \le \eta; \mathbb{R}^2)$.

Now we require $\|Ty - \underline{b}\|_{sup} \le k$ if $\|y - \underline{b}\|_{sup} \le k$.

But

$$\|Ty - \underline{b}\|_{sup} := \sup_{|x-a| \le \eta} \left\| \int_a^x f(s, y(s))\, ds \right\|_1$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x \|f(s, y(s))\|_1\, ds \right|$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x M\, ds \right| \le M\eta \le k$$

provided $M\eta \le k$ where $M = \sup_S \|f(s, y(s))\|_1$.

(exists as functions cts on bdd domain)

Finally, want T to be a contraction.

$$\|Tu - Tv\|_{sup} = \sup_{|x-a| \le \eta} \left\| \int_a^x f(s, \underline{u}(s)) - f(s, \underline{v}(s))\, ds \right\|_1$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x \|f(s, \underline{u}(s)) - f(s, \underline{v}(s))\|_1\, ds \right|$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x |f_1(s, \underline{u}(s)) - f_1(s, \underline{v}(s))| + |f_2(s, \underline{u}(s)) - f_2(s, \underline{v}(s))|\, ds \right|$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x |u_2 - v_2| + L(|u_1 - v_1| + |u_2 - v_2|)\, ds \right|$$

$$\le \sup_{|x-a| \le \eta} \left| \int_a^x (L+1)(|u_1 - v_1| + |u_2 - v_2|)\, ds \right|$$

$$\le (L+1)\eta \|\underline{u} - \underline{v}\|_1$$

So contraction provided $\eta < \frac{1}{L+1}$

Hence, if we choose

$$\eta < \min\left\{ \frac{k}{M}, \frac{1}{L+1} \right\},$$ T satisfies

conditions of CMT and has a unique fixed pt, $y(x)$, which is a solution of integral equation.

Hence $y(x)$ is cts differentiable and satisfies

S/B

$$y'(x) = f(x, y(x)), \quad y(a) = b$$ ~~[8]~~ 10

So $y(x)$ unique solution of IVP.

~~(c) We may extend the range of the solution to all $x \in [a-h, a+h]$ by iteration, Surely more detail? needed? Or just drop this part as question quite long.~~ ~~[2]~~

~~Doesn't (a)'s marks add to 12?~~

(b). Consider $y''(x) = \dfrac{y'(x)^2}{y(x)} + 2xy'(x)$

Let $y_1 = y$

$y_1' = y_2$

$$y_1' = y_2 = f_1(x, y_1, y_2)$$

$$y_2' = \frac{y_2^2}{y_1} + 2xy_2 = f_2(x, y_1, y_2)$$ [2]

Now work in the set $S = \{(x, y_1, y_2) : |x| \le h,$ [1]

$|y_1 - 1| + |y_2 - 1| \le k\}$

where $k < 1$ to ensure y_1 and y_2 never zero.

ie $-k \le y_1 - 1 \le k$, $-k \le y_2 - 1 \le k$.

So $y_1 > 0$ and $y_2'' = + \dfrac{y_2^2}{y_1} + 2xy_2$ is well

defined in $|x| \le h$. [1]

(4)

Now f_1 and f_2 are cts and hence bounded on S.

$$|f_1| = |y_2| \le k+1 = M_1$$

$$|f_2| = \left| \frac{y_2^2}{y_1} + 2xy_2 \right| \le \left| \frac{y_2^2}{y_1} \right| + |2xy_2|$$

$$\le \frac{(1+k)^2}{(1-k)} + 2h(1+k) = M_2$$

[2]

$$\text{So } M = \sup_S \|f(s, y(s))\|_1$$

$$= \sup_S \left(|f_1| + |f_2| \right) = M_1 + M_2.$$

[1]

take $M = M_1 + M_2$ where $M_1 = k+1$

$$M_2 = \frac{(1+k)^2}{(1-k)} + 2h(1+k).$$

Now f_1 and f_2 Lipschitz continuous on S

$\Longrightarrow$

$$|f_1(x, u_1, v_1) - f_1(x, u_2, v_2)| = |v_1 - v_2|$$

(see earlier)

$$\le |u_1 - u_2| + |v_1 - v_2|$$

this isn't needed - done as both of P.2

$\Rightarrow f_1$ Lipschitz continuous on S, const 1.

(no additional mark awarded if again stated here),

$$|f_2(x, u_1, v_1) - f_2(x, u_2, v_2)|$$

(showing Lipschitz cts w.r.t. u and v)

$$= \cancel{\left| \frac{v_1^2}{u_1} + 2xv_2 - \frac{v_2^2}{u_2} + 2xv_2 \right|}$$

$$= \cancel{\left| \frac{v_1^2}{u_1} - v_2^2 \right.}$$

$$= \Big| f_2(x,u_1,v_1) - f_2(x,u_1,v_2) + f_2(x,u_1,v_2) - f_2(x,u_2,v_2) \Big|$$

$$\le |f_2(x,u_1,v_1) - f_2(x,u_1,v_2)| + |f_2(x,u_1,v_2) - f_2(x,u_2,v_2)|$$ * see page 6

$$= \left| \frac{v_1^2}{u_1} + 2xv_1 - \frac{v_2^2}{u_1} - 2xv_2 \right| + \left| \frac{v_2^2}{u_1} + \cancel{2xv_2} - \frac{v_2^2}{u_2} - \cancel{2xv_2} \right|$$

$$= \left| \frac{v_1^2 - v_2^2}{u_1} + 2x(v_1 - v_2) \right| + \left| \frac{u_2 - u_1}{u_1 u_2} v_2^2 \right|$$

$$\le \frac{|v_1+v_2|}{|u_1|}|v_1-v_2| + 2|x||v_1-v_2| + \frac{|u_2-u_1|\,|v_2^2|}{|u_1 u_2|}$$

$$\le \frac{2(1+k)}{1-k}|v_1-v_2| + 2\overset{h}{\cancel{|x|}}|v_1-v_2| + \frac{(1+k)^2}{(1-k)^2}|u_2-u_1|$$

Pick $L = \sup\left\{ \frac{2(1+k)}{1-k} + 2h,\ \frac{(1+k)^2}{(1-k)^2} \right\}$

$$\le L\big(|u_1-u_2| + |v_1-v_2|\big)$$ [3]

So f_2 Lipschitz continuous

Hence there is a solution on $|x| \le \eta$ for [3]

$$\eta < \min\left(\frac{k}{M},\ \frac{1}{1+L} \right)$$

where $M = M_1 + M_2$, $L = \sup\left\{ \frac{2(1+k)}{1-k} + 2h,\ \frac{(1+k)^2}{(1-k)^2} \right\}$

$M_1 = k+1$, $M_2 = \frac{(1+k)^2}{(1-k)} + 2h(1+k)$. [15]

or use MVT in 1 dimension : (6)

$$|f_\alpha(x, u_1, v_1) - f_\alpha(x, u_1, v_2)| + |f_\alpha(x, u_1, v_2) - f_\alpha(x, u_2, v_2)|$$

$$= \left| \frac{\partial}{\partial v} f_\alpha(x, u_1, \zeta) \right| |v_1 - v_2| + \left| \frac{\partial}{\partial u} f_\alpha(x, \xi, v_2) \right| |u_1 - u_2|$$

ζ between v_1 and v_2

ξ " u_1 and u_2

$$= \left| \frac{2\zeta}{u_1} + 2x \right| |v_1 - v_2| + \left| -\frac{v_2^{2}}{\xi^2} \right| |u_1 - u_2|$$

$$\leq \left(2\frac{(1+k)}{1-k} + 2h \right) |v_1 - v_2| + \frac{(1+k)^2}{(1-k)^2} |u_1 - u_2|$$